

Semiclassical quantization of spinning strings in $\text{AdS}_4 \times \text{CP}^3$

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ABSTRACT: We derive the one-loop correction to the space-time energy of a folded string in $\text{AdS}_4 \times \text{CP}^3$ carrying spin S in AdS_4 and angular momentum J in CP^3 in the long string approximation. From this general result in the limit $J \ll \log S$ we obtain the one-loop correction to the cusp anomalous dimension which turns out to be $-\frac{5 \log 2}{2\pi}$. This value appears to be in conflict with the prediction from the recently conjectured all-loop Bethe ansatz.

KEYWORDS: AdS-CFT Correspondence, Sigma Models, Penrose limit and pp-wave background.

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1. Introduction and summary

Recently a new example of the AdS/CFT dual pair has been conjectured [1]. It involves the three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $\text{SU}(N) \times \text{SU}(N)$ and the theory of M2-branes in the eleven-dimensional geometry $\text{AdS}_4 \times S^7/\mathbb{Z}_k$, where k is the Chern-Simons level. In the scaling limit $k, N \rightarrow \infty$ with $\lambda = 2\pi^2 N/k$ fixed, the corresponding M-theory is effectively described by type IIA strings moving in the $\text{AdS}_4 \times \mathbb{CP}^3$ background.

As shown in [2, 3], the Green-Schwarz action for type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ with κ -symmetry partially fixed can be understood as the coset sigma-model on the same space supplied with a proper Wess-Zumino term.¹ Indeed, type IIA superstring involves 32 fermionic degrees of freedom (two Majorana-Weyl fermions in ten dimensions of opposite chirality); due to κ -symmetry only 16 of them are physical. On the other hand, the sigma model based on the coset space² $\text{OSP}(2, 2|6)/\text{SO}(3, 1) \times \text{U}(3)$ contains 24 fermions. However, it also exhibits κ -symmetry, which for generic backgrounds allows one to gauge away precisely 8 fermions. The remaining 16 fermions together with their bosonic partners render the physical content of $\text{AdS}_4 \times \mathbb{CP}^3$ superstring. The coset sigma model is classically

¹See also [4] for the sigma model description in the pure spinor formalism.

²The standard notation for the supergroup in the numerator of the coset is $\text{OSP}(6|4)$. We prefer, however, to use the notation $\text{OSP}(2, 2|6)$ to signify that this supergroup is an isometry group of the $\text{AdS}_4 \times \mathbb{CP}^3$ superspace. This notation is also in close analogy with $\text{PSU}(2, 2|4)$, which is an isometry of the $\text{AdS}_5 \times S^5$ superspace.

integrable [2, 3] which opens a way to investigate its dynamics in a way similar to the case of $\text{AdS}_5 \times \text{S}^5$ superstring. In particular, an algebraic curve encoding the solutions of the classical $\text{AdS}_4 \times \mathbb{CP}^3$ sigma model has been derived in [5]. Further aspects³ of classical integrability have been investigated in [7].

Complementary, the planar superconformal Chern-Simons theory appears to be integrable at leading order in the weak coupling expansion [8–10]. The corresponding Bethe equations can be embedded into those based on the supergroup $\text{OSP}(2, 2|6)$ which provides a convenient starting point to generalize them to higher loops. This, in conjunction with the knowledge of the sigma model algebraic curve and experience with the $\text{AdS}_5 \times \text{S}^5$ case [11], enabled the authors of [12] to conjecture the all-loop Bethe ansatz which should encode the anomalous dimensions of gauge theory operators (string states) for all values of λ . Some tests of the conjectured all-loop Bethe equations have been carried out in [13, 14]. Finally, the $\mathfrak{su}(2|2)$ -invariant S-matrix underlying these Bethe equations has been identified [15].

In spite of these interesting developments, the question about quantum integrability of the $\text{AdS}_4 \times \mathbb{CP}^3$ sigma model remains open. We would like to stress an apparent difference with the $\text{AdS}_5 \times \text{S}^5$ model. In the latter case the corresponding bosonic sigma model is quantum integrable and this quantum integrability extends to the whole model including fermions. In the present case, the bosonic model is integrable as well, but quantum corrections to \mathbb{CP}^3 are known to spoil its classical integrability [16]. Thus, quantum integrability of the full $\text{AdS}_4 \times \mathbb{CP}^3$ model, if exists, should essentially rely on inclusion of fermionic degrees of freedom.

An important tool to investigate the question about quantum integrability is provided by semiclassical quantization of rigid string solutions [18, 19]. Starting from a simple classical string configuration, one finds the spectrum of fluctuations around it. Summing up the fluctuation energies gives the one-loop correction to the classical energy of the spinning string which can be then compared to the value predicted by the Bethe ansatz. A particularly convenient $\text{AdS}_5 \times \text{S}^5$ solution allowing for an explicit evaluation of the one-loop energy correction is given by a rigid folded string carrying Lorentz spin S along AdS_5 . In the long string approximation the corresponding correction to the energy scales logarithmically with S and is found [18] to be

$$\delta E = -\frac{3 \log 2}{\pi} \log S.$$

On the gauge theory side, this string solution corresponds to twist two operators with large Lorentz spin S , for which the difference between the scaling dimension Δ and spin S behaves as

$$\Delta - S = f(\lambda) \log S,$$

where the function $f(\lambda)$ is the universal scaling function, also known as the cusp anomaly [20, 21]. The quantity δE provides the first correction to the strong coupling value of the cusp anomaly which has been shown [22] to perfectly agree with the Bethe

³We also point out [6], where the Penrose limit, the Landau-Lifshitz limit, the dispersion relation and giant magnon solutions both in infinite and finite volume have been studied for the $\text{AdS}_4 \times \mathbb{CP}^3$ sigma model.

ansatz prediction based on the BES equation [17]. The one-loop correction to the long (S, J) -string, which in addition to the Lorentz spin S also carries angular momentum J along a big circle of five-sphere, has been obtained in [23] and it provides the leading strong coupling correction to the so-called generalized scaling function $f(\lambda, J/\log S)$ [24, 25].

The purpose of the present paper is to perform a similar semiclassical quantization of a rigid string spinning in the $\text{AdS}_4 \times \mathbb{CP}^3$ space-time and obtain the corresponding one-loop energy shift. Namely, we consider a rigid folded string with Lorentz spin S and angular momentum J along a circle $S^1 \subset \mathbb{CP}^3$. The gauge theory operators dual to this string solution are made of two bi-fundamental scalars with S light-cone derivatives distributed among them, and they transform in the irrep $[J, 0, J]$ of $\mathfrak{su}(4)$. By finding the fluctuation spectrum around the classical solution in the long string approximation, we obtain the corresponding one-loop energy shift as a function of S and J . In particular, in the limit of "slow" rotation, $J \ll \log S$, we find that the corresponding one-loop correction is given by

$$\delta E = -\frac{5 \log 2}{2\pi} \log S.$$

Apparently, this result appears to be in contradiction with the one conjectured in [5]. According to [5], the energy correction should be half of that for the corresponding string solution on $\text{AdS}_5 \times S^5$, i.e. it should be equal to $-\frac{3 \log 2}{2\pi} \log S$. The conjecture of [5] was based on the assumption that an unknown function $h(\lambda)$ entering the all-loop Bethe ansatz has a vanishing subleading (constant) term at strong coupling. Provided we adopt the same definition of the cusp anomaly, we see that it is not the case. Clearly, further investigations are needed to clarify this important issue.

We would like to stress that our computation is a genuine field-theoretic one and it does not rely on the knowledge of the algebraic curve. It is done exclusively in the framework of the coset sigma model. As observed in [2], strings which carry only AdS spin provide an example of a singular string background, because the corresponding κ -symmetry transformations instead of generic rank 8 have higher rank equal to 12. Thus, to properly treat the fluctuation spectrum around this singular solution, we keep throughout the calculations a non-vanishing angular momentum J which can be then regarded as the regularization parameter. We find that the resulting expression for the one-loop energy shift admits a smooth limit $J \rightarrow 0$, which allows us to obtain the above stated result for the cusp anomalous dimension of high spin operators. Finally, we notice that in the limit $J \ll \log S$ the fluctuation spectrum contains 6 massless bosons and 2 massless fermions. Thus, in opposite to what happens in the $\text{AdS}_5 \times S^5$ case [26] where only 5 bosonic massless excitations are present, a would be "quantum bosonic \mathbb{CP}^3 model" is not splitting off in this limit.

The paper is organized as follows. In the next section we discuss the coset sigma model which captures the physics of type IIA strings on $\text{AdS}_4 \times \mathbb{CP}^3$. In section 3 we present the (S, J) -solution in terms of a coset element. Section 4 is devoted to the analysis of the fluctuation spectrum around the (S, J) -solution. Finally, in section 5 we compute the corresponding one-loop energy shift. Appendix A contains the details on the description of the coset space $\text{AdS}_4 \times \mathbb{CP}^3$. In appendix B we provide a detailed treatment of κ -symmetry transformations around the (S, J) -solution.

While preparing this manuscript for submission, the work [27] appeared, which seems to be in agreement with our findings. We were also informed about [28] where the same result for the one-loop energy shift was obtained.

2. String action

The sigma model describing strings on the coset space $\text{AdS}_4 \times \mathbb{CP}^3$ has been introduced in [2]. Denote by $A = -g^{-1}dg$ the flat current constructed out of a coset representative g . The sigma model action reads as

$$S = -\frac{R^2}{4\pi\alpha'} \int d\sigma d\tau \mathcal{L}, \tag{2.1}$$

where R is the radius of the AdS space and the Lagrangian density is the sum of the kinetic and the Wess-Zumino terms

$$\mathcal{L} = \gamma^{\alpha\beta} \text{str}(A_\alpha^{(2)} A_\beta^{(2)}) + \kappa \epsilon^{\alpha\beta} \text{str}(A_\alpha^{(1)} A_\beta^{(3)}). \tag{2.2}$$

Here $A^{(k)}$ denotes a homogeneous component of A of degree k under the \mathbb{Z}_4 -automorphism Ω and $\gamma^{\alpha\beta} = h^{\alpha\beta} \sqrt{-h}$ is the Weyl-invariant combination of the world-sheet metric $h_{\alpha\beta}$ with $\det \gamma = -1$. We also use the convention $\epsilon^{\tau\sigma} = 1$. To ensure κ -symmetry, the parameter κ in front of the Wess-Zumino term should be equal to ± 1 .

To proceed, one has to choose an explicit parametrization of the coset element g . We will pick

$$g = g_O g_\chi g_B, \tag{2.3}$$

where $g = e^\chi$ depends on the odd matrix χ comprising the 24 fermionic degrees of freedom of the model. The element g_O can be chosen in different ways depending on which commuting isometries we would like to be realized linearly. For instance, one can take⁴

$$g_O = \begin{pmatrix} e^{\frac{i}{2}t\Gamma^0} & 0 \\ 0 & e^{-\frac{\varphi}{2}(T_{34}+T_{56})} \end{pmatrix}, \tag{2.4}$$

where t and φ are the global AdS time and one of the angles of \mathbb{CP}^3 , respectively.⁵ Since the global symmetry group $\text{OSP}(2, 2|4)$ acts on g from the left the isometries corresponding to constant shifts of t and ϕ will be realized linearly and they do not act on the fermionic variables, i.e., fermions are unchanged under the corresponding $\text{U}(1)$ transformations [30]. We note that such a parametrization will be suitable for imposition of the uniform light-cone gauge [31].

⁴For the definition of gamma-matrices, the $\mathfrak{so}(6)$ Lie algebra generators T_{ij} , the matrices C_4, K_4, K, Υ and T_i appearing throughout the paper see appendix A, and [2].

⁵In particular, the algebra $\mathfrak{so}(3, 2)$ has rank two, so that one can choose the diagonal matrices $\frac{i}{2}\Gamma^0$ and $\frac{1}{4}[\Gamma^2, \Gamma^3]$ as the generators of commuting isometries.

Finally, the element g_B comprises all the coordinates parametrizing AdS_4 and \mathbb{CP}^3 except those which parametrize the element g_O . Explicitly,

$$g_B = \begin{pmatrix} g_{\text{AdS}} & 0 \\ 0 & g_{\mathbb{CP}} \end{pmatrix}. \quad (2.5)$$

The matrix g_{AdS} have the following characteristic properties

$$g_{\text{AdS}}^t C_4 g_{\text{AdS}} C_4^{-1} = \mathbb{I}, \quad K_4 g_{\text{AdS}} = g_{\text{AdS}}^t K_4, \quad \Gamma^5 g_{\text{AdS}} = g_{\text{AdS}}^{-1} \Gamma^5,$$

Analogously, $g_{\mathbb{CP}}$ obeys the following requirements

$$g_{\mathbb{CP}}^t g_{\mathbb{CP}} = \mathbb{I}, \quad K_6 g_{\mathbb{CP}} = g_{\mathbb{CP}}^t K_6.$$

As the consequence, the element g_B satisfies the following identity

$$\Upsilon g_B \Upsilon^{-1} = g_B^{-1}, \quad (2.6)$$

where Υ defines an inner automorphism Ω of the complexified algebra $\mathfrak{osp}(2, 2|6)$. It is worth to point out that the matrix g_O is orthosymplectic but it does not obey eq. (2.6) satisfied by the element g_B .

As was explained in [29], a convenient and compact representation of the sigma model Lagrangian can be constructed in terms of the following matrix G

$$G = \begin{pmatrix} g_{\text{AdS}} K_4 g_{\text{AdS}}^t & 0 \\ 0 & g_{\mathbb{CP}} K_6 g_{\mathbb{CP}}^t \end{pmatrix} = g_B K g_B^t. \quad (2.7)$$

By construction, this matrix is skew-symmetric: $G^t = -G$. Introducing the split

$$g_x^{-1} g_O^{-1} d(g_O g_x) = F + B, \quad (2.8)$$

where F and B are odd and even superalgebra elements, respectively, the Lagrangian (2.2) can be cast in the form [29, 2]

$$\begin{aligned} \mathcal{L} = \frac{1}{4} \text{str} & \left[\gamma^{\alpha\beta} (\partial_\alpha G G^{-1} \partial_\beta G G^{-1} + 4B_\alpha \partial_\beta G G^{-1} + 2B_\alpha B_\beta + 2B_\alpha G B_\beta^t G^{-1}) \right. \\ & \left. + 2i\kappa \epsilon^{\alpha\beta} F_\alpha G F_\beta^{\text{st}} G^{-1} \right]. \end{aligned} \quad (2.9)$$

The Lagrangian (2.9) provides a convenient starting point for studying the fluctuation spectrum around classical solutions of the string sigma model.

3. The (S, J) -string

We choose as the background solution string spinning in the directions ϕ and φ of AdS_4 and \mathbb{CP}^3 spaces, respectively. This naturally suggests to pick up as g_O the following matrix

$$g_O = \begin{pmatrix} e^{\frac{i}{2} t \Gamma^0 - \frac{1}{4} \phi [\Gamma^2, \Gamma^3]} & 0 \\ 0 & e^{-\frac{\varphi}{2} (T_{34} + T_{56})} \end{pmatrix}. \quad (3.1)$$

Then, the AdS part of the element g_B can be chosen as follows

$$g_{\text{AdS}} = e^{\frac{i}{2}\rho \sin \psi \Gamma^1 - \frac{i}{2}\rho \cos \psi \Gamma^3}. \quad (3.2)$$

Hence, in addition to the global time t , the space AdS_4 is parametrized by the non-negative variable ρ and by two angles, ϕ and ψ . As to g_{CP} , since we distinguish the angle ϕ , it is convenient to choose the remaining five coordinates on \mathbb{CP}^3 in the same way as was done in [2], namely, we parametrize g_{CP} by one real coordinate x_4 and by two complex variables y_1 and y_2 , see [2] for details. In order to keep the present discussion clear, we refer the reader to appendix A for the full details concerning the parametrization of g_{CP} .

The background solution corresponding to the (S, J) -string can be now obtained by putting to zero the AdS angle ψ together with the \mathbb{CP}^3 coordinates x_4 and y_1, y_2 , and picking up the rotating string ansatz for the remaining variables

$$t = \varkappa\tau, \quad \phi = \omega_1\tau, \quad \varphi = \omega_2\tau, \quad \rho \equiv \rho(\sigma). \quad (3.3)$$

Of course, the spinning string ansatz is embedded in the subspace $\text{AdS}_3 \times S^1$ of $\text{AdS}_4 \times \mathbb{CP}^3$, and, for this reason, the corresponding solution must coincide with the one obtained in [18]. We see that for the rotating ansatz the components of g_B reduce to

$$g_{\text{AdS}} = e^{-\frac{i}{2}\rho\Gamma^3}, \quad g_{\text{CP}} = e^{\frac{\pi}{4}T_5} \quad (3.4)$$

and, as the consequence the coset element underlying the (S, J) -string solution is of the form

$$g = \begin{pmatrix} e^{\frac{i}{2}\varkappa\tau\Gamma^0 - \frac{1}{2}\omega_1\tau\Gamma^2\Gamma^3} e^{-\frac{i}{2}\rho\Gamma^3} & 0 \\ 0 & e^{-\frac{1}{2}\omega_2\tau(T_{34}+T_{56})} e^{\frac{\pi}{4}T_5} \end{pmatrix}. \quad (3.5)$$

In the next section we will use this representation to find the Lagrangian for fluctuation modes.

Finally, we note that the parameters of the solution $(\varkappa, \omega_1, \omega_2)$ are related to the Noether charges of the model which are the space-time energy E , the AdS spin S , and the \mathbb{CP}^3 spin J as follows

$$E = \sqrt{\lambda} \varkappa \int \frac{d\sigma}{2\pi} \cosh^2 \rho, \quad S = \sqrt{\lambda} \omega_1 \int \frac{d\sigma}{2\pi} \sinh^2 \rho, \quad J = \sqrt{\lambda} \omega_2. \quad (3.6)$$

They are, of course, the same as for the (S, J) -string spinning in $\text{AdS}_3 \times S^1$. Here the parameter λ is related to the AdS radius⁶ as $\sqrt{\lambda} = \frac{R^2}{\alpha'}$.

In this paper we are mostly interested in the so-called long string limit corresponding to $\omega_1, \omega_2 \rightarrow \infty$ with the ratio $u = \frac{\omega_2}{\omega_1} = \frac{1}{\sqrt{1+x^2}}$ fixed. In this limit,

$$\kappa \approx \omega_1 \quad \text{and} \quad x = \frac{\sqrt{\lambda}}{\pi J} \ln \frac{S}{J} \quad \text{fixed}. \quad (3.7)$$

The energy of the long string is then

$$E = S + J\sqrt{1+x^2} + \dots \quad (3.8)$$

and it can be further approximated by assuming the "fast" or "slow" limits which correspond to taking $x \ll 1$ or $x \gg 1$, respectively [23].

⁶Note that λ is related to the gauge theory parameters k and N as $\lambda = 2\pi^2 N/k$.

4. Lagrangian for quadratic fluctuations

4.1 Spectrum of bosonic fluctuations

The Lagrangian for the quadratic fluctuations follows straightforwardly from the bosonic part of the action (2.9). In the conformal gauge we find

$$\begin{aligned} \mathcal{L}_B^{(2)} = & -\cosh^2 \rho \partial_\alpha \tilde{t} \partial^\alpha \tilde{t} + \sinh^2 \rho \partial_\alpha \tilde{\phi} \partial^\alpha \tilde{\phi} + 2 \sinh 2\rho \tilde{\rho} (\varkappa \partial_\tau \tilde{t} - \omega_1 \partial_\tau \tilde{\phi}) + \partial_\alpha \tilde{\rho} \partial^\alpha \tilde{\rho} + \\ & + (\varkappa^2 - \omega_1^2) \cosh 2\rho \tilde{\rho}^2 + \sinh^2 \rho (\partial_\alpha \psi \partial^\alpha \psi + \omega_1^2 \psi^2) + \partial_\alpha \tilde{\varphi} \partial^\alpha \tilde{\varphi} \\ & + \partial_\alpha x \partial^\alpha x + \omega_2^2 x^2 + \partial_\alpha v_r \partial^\alpha \bar{v}_r + \frac{\omega_2^2}{4} v_r \bar{v}_r. \end{aligned}$$

We see that the part of the action for the AdS₄ fields $\tilde{t}, \tilde{\rho}, \tilde{\phi}, \psi$ and the angular \mathbb{CP}^3 variable $\tilde{\varphi}$, shown in the first two lines, exactly agree with those in equation (5.10) of [18]. In addition we have five \mathbb{CP}^3 fields, x and two complex fields v_r . Furthermore, the linearized Virasoro constraints read

$$\frac{1}{2}(\omega_1^2 - \varkappa^2) \sinh 2\rho \tilde{\rho} - \varkappa \cosh^2 \rho \partial_\tau \tilde{t} + \omega_2 \partial_\tau \tilde{\varphi} + \omega_1 \sinh^2 \rho \partial_\tau \tilde{\phi} + \rho' \partial_\sigma \tilde{\rho} \approx 0, \quad (4.1)$$

$$\rho' \partial_\tau \tilde{\rho} - \varkappa \cosh^2 \rho \partial_\sigma \tilde{t} + \omega_2 \partial_\sigma \tilde{\varphi} + \omega_1 \sinh^2 \rho \partial_\sigma \tilde{\phi} \approx 0. \quad (4.2)$$

Obviously, the linearized Virasoro constraints are identical to equations (5.11) and (5.12) of [18]. Then, the physical fields from \mathbb{CP}^3 decouple completely from the rest. As it can be seen from the Lagrangian and also as shown in [2], these are five massive fields. In units of ω_2 , one of these fields have mass $m = 1$ and the other four $m = 1/2$.

As for the other fields, in [23] it was shown how to compute the spectrum, in the long string limit, around the solution we are interested in. According to [23], from the AdS₄ fields and $\tilde{\varphi}$ we get three physical fields. One is ψ , with mass $m_\psi^2 = 2\varkappa^2 - \omega_2^2$, while the other two modes have frequencies

$$\Omega_{\pm n}^B = \sqrt{n^2 + 2\varkappa^2 \pm 2\sqrt{\varkappa^4 + n^2 \omega_2^2}}, \quad n = 0, \pm 1, \pm 2, \dots \quad (4.3)$$

In the long string approximation $\varkappa \approx \omega_1$. Notice that in the limit $\omega_1 \gg \omega_2$, we get one field with mass (in units of ω_1) $m^2 = 4$, one field with mass $m^2 = 2$ and six massless fields, as opposed to the situation in AdS₅ × S⁵, where we get two fields of $m^2 = 2$ and five massless fields. It is these five massless fields which give rise to the O(6) sigma model in this special limit [26]. As we will see in the next section, in the present case we will also find two massless fermions in this limit. Hence the situation is pretty different to what happens in AdS₅ × S⁵.

4.2 Spectrum of fermionic fluctuations

Here we will work out the spectrum of fermionic fluctuations around the (S, J)-string. The relevant part of the Lagrangian (2.9) contributing the quadratic action for fermions is

$$\mathcal{L}_F^{(2)} = \text{str} \left[\frac{1}{2} \gamma^{\alpha\beta} B_\alpha (B_\beta + G B_\beta^t G^{-1}) + \gamma^{\alpha\beta} B_\alpha \partial_\beta G G^{-1} + \frac{i}{2} \epsilon^{\alpha\beta} F_\alpha G F_\beta^{\text{st}} G^{-1} \right]. \quad (4.4)$$

According to the formula (2.8), up to terms quadratic in fermions, we have

$$F = D\chi, \quad B = g_o^{-1}dg_o + \frac{1}{2}(D\chi\chi - \chi D\chi), \quad (4.5)$$

where we have introduced the covariant differential $D\chi = d\chi + [g_o^{-1}dg_o, \chi]$. In the conformal gauge⁷ we, therefore, find the following quadratic action

$$\begin{aligned} \mathcal{L}_F^{(2)} = & \frac{1}{2} \text{str} \left[- (g_o^{-1}\partial_\tau g_o + G(g_o^{-1}\partial_\tau g_o)^t G^{-1})(D_\tau\chi\chi - \chi D_\tau\chi) + \partial_\sigma G G^{-1}(D_\sigma\chi\chi - \chi D_\sigma\chi) \right] \\ & + \frac{i}{2} \kappa \text{str} \left[D_\tau\chi G(D_\sigma\chi)^{\text{st}} G^{-1} - D_\sigma\chi G(D_\tau\chi)^{\text{st}} G^{-1} \right], \end{aligned} \quad (4.6)$$

where we made use of the fact that g_o and G do not depend on σ and τ , respectively. Explicitly,

$$g_o^{-1}\partial_\tau g_o = \begin{pmatrix} \frac{i}{2}\varkappa\Gamma^0 - \frac{1}{2}\omega_1\Gamma^2\Gamma^3 & 0 \\ 0 & -\frac{1}{2}\omega_2(T_{34} + T_{56}) \end{pmatrix} \quad (4.7)$$

and $\partial_\sigma G G^{-1} = \text{diag}(-i\rho'\Gamma^3, 0)$.

It is clear that, in general, fermion masses will depend non-trivially on the non-constant function $\rho(\sigma)$ and its derivative which enter in the above Lagrangian through the matrix G . However, in the long string limit we are most interested in here, one can approximate $\rho'(\sigma) \approx \text{const}$. Thus, in this limit one can attempt to redefine fermions as

$$\chi \rightarrow W\chi W^{-1}, \quad (4.8)$$

where the role of W would be to remove the ρ -dependence from G . The matrix $W \in \text{OSP}(2, 2|6)$ must satisfy a few natural requirements. First, under redefinition (4.8) the covariant differential $D\chi$ undergoes the following transformation

$$D\chi \rightarrow W(d\chi + [W^{-1}g_o^{-1}dg_o W, \chi] + [W^{-1}dW, \chi])W^{-1}. \quad (4.9)$$

Thus, if we do not want to introduce an extra dependence on ρ , the matrix $W^{-1}dW$ should depend on the derivatives of ρ' only and, when being restricted to its AdS block, it should commute in the long string limit with the corresponding block of $g_o^{-1}\partial_\tau g_o$. The last requirement also guarantees that the kinetic term in eq. (4.13) will not receive an extra ρ -dependence under such redefinition of fermions. Second, W must commute with $\partial_\sigma G G^{-1}$, which is equivalent to the requirement of commuting with Γ^3 (naturally embedded into 10×10 -matrices). This will ensure that the term in the Lagrangian containing $\partial_\sigma G G^{-1}$ will not receive new ρ -dependent terms. Finally, W must be capable to remove ρ from G , i.e the element $W^{-1}G(W^t)^{-1}$ should be independent of ρ . The conditions on W stated above can be satisfied *in the long string limit only* and they fix W essentially uniquely. To construct W , we note that in the long string limit $\varkappa \approx w_1$, so that the AdS part of $g_o^{-1}\partial_\tau g_o$ becomes proportional to $i\Gamma^0 - \Gamma^2\Gamma^3$. Thus, we have to find an $\mathfrak{so}(3, 2)$ Lie algebra element, such that it commutes with two matrices

$$i\Gamma^0 - \Gamma^2\Gamma^3 \quad \text{and} \quad \Gamma^3.$$

⁷We take $\gamma^{\tau\tau} = -1 = -\gamma^{\sigma\sigma}$ and $\gamma^{\tau\sigma} = 0$.

One can easily see that the corresponding element is given by

$$i\Gamma^3 - \Gamma^0\Gamma^2.$$

Here Γ^3 is the Lie algebra coset representative, while $[\Gamma^0, \Gamma^2]$ belongs to the stability subalgebra $\mathfrak{so}(3, 1)$. The last observation implies that taking W in the form

$$W = \begin{pmatrix} e^{-\frac{\rho}{2}(i\Gamma^3 - \Gamma^0\Gamma^2)} & 0 \\ 0 & e^{\frac{\pi}{4}T_5} \end{pmatrix}, \quad (4.10)$$

we satisfy all the requirements stated, getting, in particular,

$$W^{-1}G(W^{-1})^t = K,$$

where the matrix K is defined by eq. (A.7). Since $e^{-\frac{\pi}{4}T_5}(T_{34} + T_{56})e^{\frac{\pi}{4}T_5} = -T_6$, we see that after redefining the fermions by W , the covariant derivative (4.9) acquires the following form

$$D_\alpha = \partial_\alpha + [Q_\alpha, \dots],$$

where the composite vector field Q_α has the components

$$Q_\tau = \text{diag}\left(Q_\tau^{\text{AdS}}, -\frac{1}{2}\omega_2 T_6\right), \quad Q_\sigma = W^{-1}\partial_\sigma W, \quad (4.11)$$

where

$$Q_\tau^{\text{AdS}} = \frac{1}{4}(\varkappa + \omega_1)\left(i\Gamma^0 - \Gamma^2\Gamma^3\right) + \frac{1}{4}(\varkappa - \omega_1)\left[\cosh 2\rho\left(i\Gamma^0 + \Gamma^2\Gamma^3\right) + \sinh 2\rho\left(i\Gamma^2 + \Gamma^0\Gamma^3\right)\right]. \quad (4.12)$$

In the long string limit $\varkappa \approx \omega_1$ the function ρ drops out of Q_τ as it should be. Also, by construction, in the long string limit the commutator $[Q_\alpha, Q_\beta]$ vanishes, *i.e.* the connection D_α becomes flat.

We further note that since $\chi \in \mathfrak{osp}(2, 2|6)$ its supertranspose is $\chi^{\text{st}} = -C\chi C^{-1}$ and, therefore, after the redefinition of fermions has been done, the action (4.6) can be cast in the form

$$\begin{aligned} \mathcal{L}_F^{(2)} = & \frac{1}{2} \text{str} \left[-(Q_\tau - \Upsilon Q_\tau \Upsilon^{-1})(D_\tau \chi \chi - \chi D_\tau \chi) + \partial_\sigma G G^{-1}(D_\sigma \chi \chi - \chi D_\sigma \chi) \right] + \\ & + \frac{i}{2} \kappa \text{str} \left[D_\tau \chi \Upsilon D_\sigma \chi \Upsilon^{-1} - D_\sigma \chi \Upsilon D_\tau \chi \Upsilon^{-1} \right]. \end{aligned} \quad (4.13)$$

Note that the kinetic term for fermions is projected on the space $\mathcal{A}^{(2)}$ as $Q_\tau - \Upsilon Q_\tau \Upsilon^{-1} \in \mathcal{A}^{(2)}$. In particular, in the long string limit

$$Q_\tau - \Upsilon Q_\tau \Upsilon^{-1} \approx i\varkappa\Gamma^0 + \omega_2 T_6.$$

One can check that for a generic χ the kinetic term of the Lagrangian (4.13) is degenerate and it has rank 16. This is a manifestation of κ -symmetry which allows one to remove 8

unphysical fermions out of 24 making thereby the kinetic term non-degenerate [2]. As is shown in appendix B, an admissible and convenient κ -symmetry gauge choice is

$$\theta T_{56} = 0, \tag{4.14}$$

which removes the fermions from the fifth and the sixth column of χ .

Introducing a 4 by 4 matrix ϑ made of non-vanishing entries of θ , we can write the quadratic κ -gauge fixed Lagrangian in the long string approximation as follows

$$\begin{aligned} \mathcal{L}_F^{(2)} = & -\varkappa \operatorname{tr}(\vartheta^t \Gamma^3 \dot{\vartheta}) + \frac{\varkappa^2}{2} \operatorname{tr} \left[\vartheta^t C_4 (\mathbb{I} + i\Gamma^0 \Gamma^2 \Gamma^3) \vartheta \right] + \frac{\omega_2^2}{4} \operatorname{tr} (\mathbb{I} - \Gamma^0) \vartheta^t C_4 \vartheta \\ & - \rho' \operatorname{tr}(\vartheta^t \Gamma^0 \vartheta') + \frac{\rho'^2}{2} \operatorname{tr} \left[\vartheta^t C_4 (\mathbb{I} - i\Gamma^0 \Gamma^2 \Gamma^3) \vartheta \right] \\ & + i\kappa\rho' \operatorname{tr}(\vartheta^t \Gamma^0 \Gamma^5 \dot{\vartheta} K_4) + i\kappa\varkappa \operatorname{tr}(\vartheta^t \Gamma^3 \Gamma^5 \vartheta' K_4) + \kappa\varkappa\rho' \operatorname{tr} \left[\vartheta^t (\mathbb{I} - i\Gamma^0 \Gamma^2 \Gamma^3) \Gamma^5 \vartheta K_4 \right]. \end{aligned} \tag{4.15}$$

One can check that this action is hermitian provided the fermions satisfy the reality condition $\vartheta^\dagger = i\vartheta^t \Gamma^3$. Introducing the Dirac conjugate $\bar{\vartheta} = \vartheta^\dagger \Gamma^0 = i\vartheta^t \Gamma^0 \Gamma^3 = -\vartheta^t C_4$, we recognize that the reality condition is nothing else but the Majorana condition.

To compute the one-loop energy shift, one has first to determine the spectrum of fermion frequencies from the quadratic action (4.15). This is essentially the same as to solve the corresponding equations of motion. Every solution will be characterized by the energy k_0 and the momentum k_1 . Then, every zero eigenvalue of the quadratic form defining (4.15) will give a (dispersion) relation between k_0 and k_1 , while the corresponding eigenstate will be a solution of the equations of motion. Thus, we may look for the spectrum of the model by requiring that the determinant of the corresponding quadratic form is zero. There are as many particles in the theory as there are linearly independent solutions.

This is precisely the strategy we would like to follow in this paper, therefore let us discuss in more detail some subtleties which we encounter on our way. Combining the fermions ϑ in one 16-dimensional vector, the action implied by (4.15) can be schematically written as

$$S = -\frac{R^2}{4\pi\alpha'} \int d\sigma d\tau (\vartheta_i K_\tau^{ij} \partial_\tau \vartheta_j + \vartheta_i K_\sigma^{ij} \partial_\sigma \vartheta_j + \vartheta_i M^{ij} \vartheta_j). \tag{4.16}$$

In our treatment we will impose the reality condition on ϑ only at the end of calculation, i.e. we prefer to start with the action above, where in each term we have two ϑ 's rather than ϑ and ϑ^* . Varying the action, we get

$$\delta S = -\frac{R^2}{4\pi\alpha'} \int d\sigma d\tau \left(\delta\vartheta_i \widehat{K}_\tau^{ij} \partial_\tau \vartheta_j + \delta\vartheta_i \widehat{K}_\sigma^{ij} \partial_\sigma \vartheta_j + \delta\vartheta_i \widehat{M}^{ij} \vartheta_j \right),$$

where we have used anti-commutativity of fermions and integration by parts. Here

$$\begin{aligned} \widehat{K}_\tau &= K_\tau + K_\tau^t, \\ \widehat{K}_\sigma &= K_\sigma + K_\sigma^t, \\ \widehat{M} &= M - M^t. \end{aligned}$$

Thus, equations for motion look as

$$(\widehat{K}_\tau \partial_\tau + \widehat{K}_\sigma \partial_\sigma + \widehat{M})\theta = 0. \tag{4.17}$$

In momentum space the equation above yields

$$(i\widehat{K}_\tau k_0 + i\widehat{K}_\sigma k_1 + \widehat{M})\theta = 0. \quad (4.18)$$

As it follows from the discussion above, the spectrum of the model is determined by the condition

$$\mathcal{D} = \det \left[i\widehat{K}_\tau k_0 + i\widehat{K}_\sigma k_1 + \widehat{M} \right] = 0, \quad (4.19)$$

where \widehat{K}_τ and \widehat{K}_σ are symmetric matrices, and \widehat{M} is antisymmetric. We view (4.19) as an algebraic equation for k_0 , and its roots (as functions of k_1) give us the dispersion relations for all particles in the theory.

Computing the determinant, we find

$$\begin{aligned} \mathcal{D} = & 2^8 \omega_2^{16} [(2k_0 - \omega_2)^2 - 4(k_1^2 + \varkappa^2)]^2 [(2k_0 + \omega_2)^2 - 4(k_1^2 + \varkappa^2)]^2 \times \\ & \times [k_0^4 - k_0^2(2k_1^2 + \varkappa^2) + k_1^2(k_1^2 - \omega_2^2 + \varkappa^2)]^2. \end{aligned} \quad (4.20)$$

Setting $k_1 \equiv n \in \mathbb{Z}$, due to the fact that this momentum corresponds to the compact σ direction of the string world-sheet, yields the following result for the fermionic frequencies (counting given in terms of elementary fermions, rather than Majorana sets of fermions):

- 2 fermions with frequency $\frac{\omega_2}{2} + \sqrt{n^2 + \varkappa^2}$
- 2 fermions with frequency $-\frac{\omega_2}{2} + \sqrt{n^2 + \varkappa^2}$
- 2 fermions with frequency $\sqrt{n^2 + \frac{1}{2}\varkappa^2 + \frac{1}{2}\sqrt{\varkappa^4 + 4\omega_2^2 n^2}}$
- 2 fermions with frequency $\sqrt{n^2 + \frac{1}{2}\varkappa^2 - \frac{1}{2}\sqrt{\varkappa^4 + 4\omega_2^2 n^2}}$

plus the other eight fermions whose frequencies are equal to the above with negative sign. The reality condition then implies that these negative frequency fermions are nothing else but the conjugate momenta for the positive frequency ones. The constant shifts by $\pm\omega_2/2$ in the first four frequencies can be removed by an extra time-dependent redefinition of fermions, similar to that done in [2]. The resulting dispersion relation is the same as for relativistic fermions with the mass $m^2 = \varkappa^2$. In any case, even without doing this field redefinition, the shifts by $\pm\omega_2/2$ are cancelled out in the one-loop energy correction. In the special limit $\varkappa \approx \omega_1 \gg \omega_2$ the spectrum above will contain two massless fermions.

5. One-loop energy shift

Having found the frequency modes of bosons and fermions, we can readily compute the one-loop correction to the energy of the long (S, J) -string. This computation is very similar to that of [23]. The one-loop correction to the energy is given by the following sum

$$\begin{aligned} \delta E = & \frac{1}{\omega_1} \sum_{n=1}^{\infty} \left[\left(\Omega_{+,n}^B + \Omega_{-,n}^B + \sqrt{n^2 + 2\omega_1^2 - \omega_2^2} + \sqrt{n^2 + \omega_2^2} + 4\sqrt{n^2 + \frac{\omega_2^2}{4}} \right) - \right. \\ & \left. - \left(2\Omega_{+,n}^F + 2\Omega_{-,n}^F + 4\sqrt{n^2 + \omega_1^2} \right) \right], \end{aligned} \quad (5.1)$$

where

$$\Omega_{\pm,n}^F = \sqrt{n^2 + \frac{\omega_1^2}{2} \pm \frac{1}{2}\sqrt{\omega_1^4 + 4\omega_2^2 n^2}}, \quad (5.2)$$

are the non-trivial fermionic frequencies found in the previous section. It is gratifying to see that the divergencies of bosons cancel against those of fermions, so that the sum (5.1) is convergent.

We are most interested in the value of the sum in the scaling limit, $\omega_1, \omega_2 \rightarrow \infty$ with $u = \frac{\omega_2}{\omega_1}$ fixed. Following [23], in this limit, the sum can be replaced by an integral

$$\delta E = \omega_1 \int_0^\infty dp \left[\left(\Omega_+^B(p) + \Omega_-^B(p) + \sqrt{p^2 + 2 - u^2} + \sqrt{p^2 + u^2} + 4\sqrt{p^2 + \frac{u^2}{4}} \right) - \left(2\Omega_+^F(p) + 2\Omega_-^F(p) + 4\sqrt{p^2 + 1} \right) \right], \quad (5.3)$$

where

$$\Omega_{\pm}^B(p) = \sqrt{p^2 + 2 \pm 2\sqrt{1 + p^2 u^2}}, \quad \Omega_{\pm}^F(p) = \sqrt{p^2 + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4p^2 u^2}}. \quad (5.4)$$

The simplest way to compute the integral is to impose a cut-off and send it to infinity at the end of the computation. The integrals involving Ω^B and Ω^F can be simplified by using the identities

$$\begin{aligned} \Omega_+^B(p) + \Omega_-^B(p) &= \sqrt{4u^2 + (p + \sqrt{p^2 + 4 - 4u^2})^2}, \\ \Omega_+^F(p) + \Omega_-^F(p) &= \frac{1}{2}\sqrt{4u^2 + (2p + \sqrt{4p^2 + 4 - 4u^2})^2}. \end{aligned}$$

Notice that these two expressions are related as

$$\Omega_+^B(p) + \Omega_-^B(p) = 2\Omega_+^F(p/2) + 2\Omega_-^F(p/2).$$

The integrals can then be easily performed by making the change of variables $p \rightarrow q = p + \sqrt{p^2 + 4 - 4u^2}$. Finally, we obtain

$$\begin{aligned} \delta E = \frac{\omega_1}{4} &\left[-2u^2 \log u^2 - 2 \log(8 - 4u^2) + u^2 \log(16(2 - u^2)) + \right. \\ &\left. + 2 \left(-1 + u^2 + \sqrt{1 - u^2} + (u^2 - 2) \log(1 + \sqrt{1 - u^2}) \right) \right]. \quad (5.5) \end{aligned}$$

This formula describing the one-loop correction to the classical energy of the long (S, J) -string is our main result. It has to be compared with a corresponding result for the (S, J) -string spinning in $\text{AdS}_5 \times S^5$ given by eq. (2.29) in [23]. Curiously enough, we find that

$$\delta E^{\text{AdS}_4 \times \text{CP}^3} - \frac{1}{2} \delta E^{\text{AdS}_5 \times S^5} = \omega_1 (u^2 - 1) \log 2. \quad (5.6)$$

This result is in apparent contradiction to the conjecture made in [12]. According to their claim, the r.h.s. of eq. (5.6) should vanish. In the $u \rightarrow 0$ limit we obtain

$$\delta E = -\frac{5}{2} \omega_2 x \log 2 = -\frac{5 \log 2}{2\pi} \log \frac{S}{J}. \quad (5.7)$$

The coefficient in front of $\log \frac{S}{J}$ should be interpreted as the one loop correction to the cusp anomalous dimension.

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A. The coset space $\text{AdS}_4 \times \mathbb{CP}^3$

To make the paper self-contained, in this appendix we recapitulate the basic facts about the description of the coset space $\text{AdS}_4 \times \mathbb{CP}^3 = \text{OSP}(2, 2|6)/\text{SO}(3, 1) \times \text{U}(3)$.

As in [2], we introduce the following gamma-matrices

$$\begin{aligned} \Gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \Gamma^1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \Gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^3 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{A.1}$$

satisfying the Clifford algebra relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is Minkowski metric with signature $(1, -1, -1, -1)$. We also define

$$\Gamma^5 = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3, \quad C_4 = i\Gamma^0\Gamma^3, \quad K_4 = -\Gamma^1\Gamma^2. \tag{A.2}$$

In particular, C_4 is the charge conjugation matrix: $(\Gamma^\mu)^t = -C_4\Gamma^\mu C_4$, while K_4 satisfies $(\Gamma^\mu)^t = K_4\Gamma^\mu K_4^{-1}$. One has

$$(\Gamma^5)^2 = \mathbb{I}, \quad K_4^2 = -\mathbb{I}, \quad C_4^2 = -\mathbb{I}, \quad \Gamma^5 = K_4 C_4.$$

The generators $\frac{1}{4}[\Gamma^\mu, \Gamma^\nu]$ span the algebra $\mathfrak{so}(3, 1) \sim \mathfrak{usp}(2, 2)$. Adding to this set of generators the four matrices $\frac{i}{2}\Gamma^\mu$, one obtains a realization of $\mathfrak{so}(3, 2)$.

Consider 10×10 supermatrices

$$A = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix}, \tag{A.3}$$

where X and Y are even (bosonic) 4×4 and 6×6 matrices, respectively. The 4×6 matrix θ and the 6×4 matrix η are odd, i.e. linear in fermionic variables. As the matrix

superalgebra, the Lie superalgebra $\mathfrak{osp}(2, 2|6)$, is spanned by supermatrices A satisfying two conditions

$$A^{\text{st}} = -CAC^{-1}, \quad A^\dagger = -\Gamma A \Gamma^{-1}, \quad (\text{A.4})$$

where $C = \text{diag}(C_4, \mathbb{I}_6)$ and $\Gamma = \text{diag}(\Gamma^0, -\mathbb{I}_6)$. Here A^{st} stands for the supertranspose of A :

$$A^{\text{st}} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix}. \quad (\text{A.5})$$

The bosonic subalgebra of $\mathfrak{osp}(2, 2|6)$ is $\mathfrak{usp}(2, 2) \oplus \mathfrak{so}(6)$. Explicitly, the fermionic matrices obey

$$\eta = -\theta^t C_4, \quad \theta^* = i\Gamma^3 \theta, \quad (\text{A.6})$$

i.e. fermions are symplectic Majorana with the total number of real fermionic components equal to 24.

We further introduce a 6×6 matrix K_6 and a 10×10 matrix $K = \text{diag}(K_4, K_6)$:

$$K_6 = \mathbb{I}_3 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \mathbb{I}_5 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.7})$$

This matrices can be used to define an automorphism Ω of order four of the complexified algebra $\mathfrak{osp}(4|6)$

$$\Omega(A) = \begin{pmatrix} K_4 X^t K_4 & K_4 \eta^t K_6 \\ -K_6 \theta^t K_4 & K_6 Y^t K_6 \end{pmatrix} = -\Sigma K A^{\text{st}} K^{-1} \Sigma^{-1}. \quad (\text{A.8})$$

Here $\Sigma = \text{diag}(\mathbb{I}_4, -\mathbb{I}_6)$ is the grading matrix. The orthosymplectic condition for A implies

$$\Omega(A) = (\Sigma K C) A (\Sigma K C)^{-1} \equiv \Upsilon A \Upsilon^{-1}, \quad (\text{A.9})$$

i.e. Ω is an inner automorphism. Explicitly,

$$\Upsilon = \begin{pmatrix} \Gamma^5 & 0 \\ 0 & -K_6 \end{pmatrix}. \quad (\text{A.10})$$

As the vector space, $\mathcal{A} = \mathfrak{osp}(2, 2|6)$ can be decomposed under Ω into the direct sum of homogeneous components: $\mathcal{A} = \sum_{k=0}^3 \mathcal{A}^{(k)}$, where the projection $A^{(k)}$ of a generic element $A \in \mathfrak{osp}(2, 2|6)$ on the subspace $\mathcal{A}^{(k)}$ is define as

$$A^{(k)} = \frac{1}{4} \left(A + i^{3k} \Omega(A) + i^{2k} \Omega^2(A) + i^k \Omega^3(A) \right). \quad (\text{A.11})$$

In particular, $\mathcal{A}^{(0)} = \mathfrak{so}(3, 1) \oplus \mathfrak{u}(3)$.

Throughout the paper we use the generators T_{ij} of $\mathfrak{so}(6)$ defined as $T_{ij} = E_{ij} - E_{ji}$, where E_{ij} are the standard matrix unities. We also introduce the following six matrices T_6 which are Lie algebra generators of $\mathfrak{so}(6)$ along the \mathbb{CP}^3 directions:

$$\begin{aligned} T_1 &= E_{13} - E_{31} - E_{24} + E_{42}, & T_2 &= E_{14} - E_{41} + E_{23} - E_{32}, \\ T_3 &= E_{15} - E_{51} - E_{26} + E_{62}, & T_4 &= E_{16} - E_{61} + E_{25} - E_{52}, \\ T_5 &= E_{35} - E_{53} - E_{46} + E_{64}, & T_6 &= E_{36} - E_{63} + E_{45} - E_{54}. \end{aligned} \tag{A.12}$$

These generators are normalized as $\text{tr}(T_i T_j) = -4\delta_{ij}$.

According to [2], a generic $\text{SO}(6)$ element parametrizing the coset space $\mathbb{CP}^3 = \text{SO}(6)/\text{U}(3)$ can be written as

$$g_{\mathbb{CP}^3} = e^{y_i T_i}. \tag{A.13}$$

We parametrize \mathbb{CP}^3 by means of the spherical coordinates $(r, \varphi, \theta, \alpha_1, \alpha_2, \alpha_3)$, or, alternatively, by means of three complex inhomogeneous coordinate w_i ,

$$\begin{aligned} y_1 + iy_2 &= r \sin \theta \cos \frac{\alpha_1}{2} e^{\frac{i}{2}(\alpha_2 + \alpha_3) + \frac{i}{2}\varphi} = \frac{r}{|w|} w_1, \\ y_3 + iy_4 &= r \sin \theta \sin \frac{\alpha_1}{2} e^{-\frac{i}{2}(\alpha_2 - \alpha_3) + \frac{i}{2}\varphi} = \frac{r}{|w|} w_2, \\ y_5 + iy_6 &= r \cos \theta e^{i\varphi} = \frac{r}{|w|} w_3, \end{aligned} \tag{A.14}$$

where $|w|^2 = \bar{w}_k w_k$ and $\sin r = \frac{|w|}{\sqrt{1+|w|^2}}$. The geodesic circle described by the angle φ corresponds to taking $\theta = 0$ and $r = \frac{\pi}{4}$, or, equivalently, $w_3 = e^{i\varphi}$ and $w_1 = 0 = w_2$. If we further extract a geodesic angle φ by introduce one real field x and two complex fields v_1 and v_2 :

$$w_3 = (1-x)e^{i\varphi}, \quad w_1 = \frac{1}{\sqrt{2}} v_1 e^{i\varphi/2}, \quad w_2 = \frac{1}{\sqrt{2}} v_2 e^{i\varphi/2}, \tag{A.15}$$

then the corresponding quadratic action for the \mathbb{CP}^3 fluctuation modes around the (S, J) -string solution coincides with the plane-wave action obtained in [2].

B. Kappa-symmetry

Here we present an independent analysis of κ -symmetry transformations in the background of the (S, J) -string. As was explained in [2], κ -symmetry acts on the coset element by multiplication from the right:

$$g \rightarrow g e^\epsilon = g' g_c, \tag{B.1}$$

where g_c is a compensating group element from the denominator of the coset. We see that at linear order in χ and ϵ we get

$$g \rightarrow g_O g_\chi g_B e^\epsilon = g_O e^{\chi} e^{g_B \epsilon g_B^{-1}} g_B \approx g_O e^{\chi + g_B \epsilon g_B^{-1}} g_B \tag{B.2}$$

Thus, at the linearized level the fermion matrix χ changes under the κ -symmetry variation as

$$\chi \rightarrow \chi + g_B \epsilon g_B^{-1}. \quad (\text{B.3})$$

Note also that the compensation matrix g_c which depends on the even number of fermions does not arise for the linearized transformations. The parameter $\epsilon = \epsilon^{(1)} + \epsilon^{(2)}$ in the above formula is the one found in [2], e.g.,

$$\epsilon^{(1)} = A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} \kappa_{++}^{\alpha\beta} + \kappa_{++}^{\alpha\beta} A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} + A_{\alpha,-}^{(2)} \kappa_{++}^{\alpha\beta} A_{\beta,-}^{(2)} - \frac{1}{8} \text{str}(\Upsilon^2 A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)}) \kappa_{++}^{\alpha\beta}, \quad (\text{B.4})$$

where $\kappa_{++}^{\alpha\beta}$ is the κ -symmetry parameter.⁸ It is easy to find

$$A_{\tau}^{(2)} = -\frac{1}{2} g_B^{-1} \left(g_O^{-1} \partial_{\tau} g_O + G (g_O^{-1} \partial_{\tau} g_O)^t G^{-1} \right) g_B, \quad (\text{B.5})$$

$$A_{\sigma}^{(2)} = -\frac{1}{2} g_B^{-1} (\partial_{\sigma} G G^{-1}) g_B. \quad (\text{B.6})$$

Hence, in the conformal gauge

$$A_{\tau,-}^{(2)} = \frac{1}{2} (A_{\tau}^{(2)} - A_{\sigma}^{(2)}) \equiv g_B^{-1} \hat{A} g_B, \quad (\text{B.7})$$

where

$$\hat{A} = -\frac{1}{4} \left(g_O^{-1} \partial_{\tau} g_O + G (g_O^{-1} \partial_{\tau} g_O)^t G^{-1} - \partial_{\sigma} G G^{-1} \right). \quad (\text{B.8})$$

An element G entering the last formula is determined from eq. (2.7) to be

$$G = \begin{pmatrix} e^{-\frac{i}{2}\rho\Gamma^3} K_4 e^{-\frac{i}{2}\rho\Gamma^3} & 0 \\ 0 & e^{\frac{\pi}{4}T_5} K_6 e^{-\frac{\pi}{4}T_5} \end{pmatrix} = \begin{pmatrix} e^{-i\rho\Gamma^3} K_4 & 0 \\ 0 & e^{\frac{\pi}{2}T_5} K_6 \end{pmatrix}. \quad (\text{B.9})$$

We also note that since we pulled out the factor g_B out of $A^{(2)}$, the matrix \hat{A} is not element of the space $\mathcal{A}^{(2)}$.

Thus, under κ -symmetry transformation the fermionic matrix χ is shifted by

$$g_B \epsilon^{(1)} g_B^{-1} = \hat{A} \hat{A} \kappa + \kappa \hat{A} \hat{A} + \hat{A} \kappa \hat{A} - \frac{1}{8} \text{str}(\Upsilon^2 \hat{A} \hat{A}) \kappa.$$

In order find out implementation of this formula for χ , we have to understand the structure of the matrix \hat{A} . Calculations reveal the following remarkably simple formula

$$\hat{A} = \frac{\omega_2}{4} \begin{pmatrix} -\frac{i}{\omega_2} e^{-\frac{i}{2}\rho\Gamma^3} (\varkappa \cosh \rho\Gamma^0 - \omega_1 \sinh \rho\Gamma^2 + \rho'\Gamma^3) e^{\frac{i}{2}\rho\Gamma^3} & 0 \\ 0 & T_{34} + T_{56} \end{pmatrix}.$$

The non-trivial Virasoro constraint written in terms of \hat{A} implies

$$0 = 4 \text{str}(\hat{A} \hat{A}) = \rho'^2 + \omega_1^2 \sinh^2 \rho - \varkappa^2 \cosh^2 \rho + \omega_2^2, \quad (\text{B.10})$$

⁸We present the complete analysis for $\epsilon^{(1)}$ only, the computation of $\epsilon^{(2)}$ goes along the same lines.

which is an equation for the function ρ . An important about the matrix \hat{A} is that it is not constant on the world-sheet, quite in opposite to the point-particle case. On the other hand, since an expression

$$\varkappa \cosh \rho \Gamma^0 - \omega_1 \sinh \rho \Gamma^2 + \rho' \Gamma^3 \quad (\text{B.11})$$

multiplied with i takes values in $\mathcal{A}^{(2)}$, one can always find a similarity transformation with an element V from $\text{SO}(3, 1)$, which brings \hat{A} to a constant matrix, e.g, to Γ^0 , namely,

$$\varkappa \cosh \rho \Gamma^0 - \omega_1 \sinh \rho \Gamma^2 + \rho' \Gamma^3 = \omega_2 V \Gamma^0 V^{-1}, \quad (\text{B.12})$$

where we have taken into account that on solutions of the Virasoro constraint (B.10), the eigenvalues of matrix (B.11) are $\pm \omega_2$. For instance, one can take

$$V = v \begin{pmatrix} \omega_1 \sinh \rho - i\rho' & 0 & 0 & \omega_2 - \kappa \cosh \rho \\ 0 & \omega_1 \sinh \rho + i\rho' & \omega_2 - \kappa \cosh \rho & 0 \\ 0 & \omega_2 - \kappa \cosh \rho & \omega_1 \sinh \rho - i\rho' & 0 \\ \omega_2 - \kappa \cosh \rho & 0 & 0 & \omega_1 \sinh \rho + i\rho' \end{pmatrix}, \quad (\text{B.13})$$

where the unessential normalization constant is fixed by requiring $\det V = 1$. Indeed, one can check that on solutions of the Virasoro constraint the relation (B.12) is satisfied. Thus, the matrix \hat{A} exhibits the following factorizable structure

$$\hat{A} = \frac{\omega_2}{4} \mathcal{V} \mathcal{A} \mathcal{V}^{-1}, \quad (\text{B.14})$$

where we have introduced two matrices:

$$\mathcal{A} = \begin{pmatrix} -i\Gamma^0 & 0 \\ 0 & T_{34} + T_{56} \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} e^{-\frac{i}{2}\rho\Gamma^3} V & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad (\text{B.15})$$

where, in particular, matrix \mathcal{A} does not depend on the world-sheet variables. We thus see that under a linearized κ -symmetry transformation the combination $\mathcal{V}^{-1} \chi \mathcal{V}$ undergoes a shift by an element

$$\frac{\omega^2}{16} \left[\mathcal{A}^2 (\mathcal{V}^{-1} \kappa \mathcal{V}) + \mathcal{A} (\mathcal{V}^{-1} \kappa \mathcal{V}) \mathcal{A} + (\mathcal{V}^{-1} \kappa \mathcal{V}) \mathcal{A}^2 - \frac{1}{8} \text{str}(\Upsilon^2 \mathcal{A}^2) (\mathcal{V}^{-1} \kappa \mathcal{V}) \right]. \quad (\text{B.16})$$

An easy calculation shows that the matrix above has a structure

$$\frac{\omega_2^2}{16} \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^t C_4 & 0 \end{pmatrix}, \quad (\text{B.17})$$

where ε the matrix ε depends on 8 fermions only, i.e. the rank of the on-shell κ -symmetry transformations is equal to eight, confirming thereby the conclusions of [2]. Thus, our analysis shows that transformation (B.17) suffices to gauge away from the general element

$$\mathcal{V}^{-1} \chi \mathcal{V} = \begin{pmatrix} 0 & V^{-1} e^{-\frac{i}{2}\rho\Gamma^3} \theta \\ -(V^{-1} e^{-\frac{i}{2}\rho\Gamma^3} \theta)^t C_4 & 0 \end{pmatrix} \quad (\text{B.18})$$

precisely eight fermions.

Finally, we note that in section (4.2) we made an additional rotation of χ with the matrix W given by eq. (4.10). To find how the new fermionic matrix transforms under κ -symmetry, we have to rotate the parameter κ in the same way $\kappa \rightarrow W\kappa W^{-1}$. In the \mathbb{CP}^3 sector this rotation effectively leads to modifying the matrix \mathcal{A} in the following way

$$\mathcal{A} \rightarrow \mathcal{A} = \begin{pmatrix} -i\Gamma^0 & 0 \\ 0 & T_6 \end{pmatrix},$$

which is the consequence of $e^{\frac{\pi}{4}T_5}(T_{34} + T_{56})e^{-\frac{\pi}{4}T_5} = T_6$. This new matrix \mathcal{A} coincides with the one used in the paper [2], where it was concluded that the corresponding κ -symmetry transformations allow one to make the gauge choice

$$\theta T_{56} = 0,$$

which puts to zero the fifth and the sixth column of θ .

References

- [1] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *N = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)].
- [2] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a Coset σ -model*, *JHEP* **09** (2008) 129 [[arXiv:0806.4940](#)].
- [3] B.J. Stefanski, *Green-Schwarz action for type IIA strings on $AdS_4 \times CP^3$* , *Nucl. Phys. B* **808** (2009) 80 [[arXiv:0806.4948](#)].
- [4] P. Fré and P.A. Grassi, *Pure spinor formalism for $Osp(N|4)$ backgrounds*, [arXiv:0807.0044](#).
- [5] N. Gromov and P. Vieira, *The AdS_4/CFT_3 algebraic curve*, [arXiv:0807.0437](#).
- [6] G. Grignani, T. Harmark and M. Orselli, *The $SU(2) \times SU(2)$ sector in the string dual of $N = 6$ superconformal Chern-Simons theory*, [arXiv:0806.4959](#);
G. Grignani, T. Harmark, M. Orselli and G.W. Semenoff, *Finite size giant magnons in the string dual of $N = 6$ superconformal Chern-Simons theory*, [arXiv:0807.0205](#).
- [7] B. Chen and J.-B. Wu, *Semi-classical strings in $AdS_4 \times CP^3$* , *JHEP* **09** (2008) 096 [[arXiv:0807.0802](#)];
B.-H. Lee, K.L. Panigrahi and C. Park, *Spiky strings on $AdS_4 \times CP^3$* , [arXiv:0807.2559](#);
C. Ahn, P. Bozhilov and R.C. Rashkov, *Neumann-Rosochatius integrable system for strings on $AdS_4 \times CP^3$* , *JHEP* **09** (2008) 017 [[arXiv:0807.3134](#)].
- [8] J.A. Minahan and K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons*, *JHEP* **09** (2008) 040 [[arXiv:0806.3951](#)].
- [9] D. Gaiotto, S. Giombi and X. Yin, *Spin chains in $N = 6$ superconformal Chern-Simons-matter theory*, [arXiv:0806.4589](#).
- [10] D. Bak and S.-J. Rey, *Integrable spin chain in superconformal Chern-Simons theory*, *JHEP* **10** (2008) 053 [[arXiv:0807.2063](#)].

- [11] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [[hep-th/0402207](#)];
 G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)];
 N. Beisert and M. Staudacher, *Long-range PSU(2,2|4) Bethe ansätze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [[hep-th/0504190](#)].
- [12] N. Gromov and P. Vieira, *The all loop AdS₄/CFT₃ Bethe ansatz*, [arXiv:0807.0777](#).
- [13] D. Astolfi, V.G.M. Puletti, G. Grignani, T. Harmark and M. Orselli, *Finite-size corrections in the SU(2) × SU(2) sector of type IIA string theory on AdS₄ × CP³*, [arXiv:0807.1527](#).
- [14] I. Shenderovich, *Giant magnons in AdS₄/CFT₃: dispersion, quantization and finite-size corrections*, [arXiv:0807.2861](#).
- [15] C. Ahn and R.I. Nepomechie, *N = 6 super Chern-Simons theory S-matrix and all-loop Bethe ansatz equations*, *JHEP* **09** (2008) 010 [[arXiv:0807.1924](#)].
- [16] E. Abdalla, M. Forger and M. Gomes, *On the origin of anomalies in the quantum nonlocal charge for the generalized nonlinear σ-models*, *Nucl. Phys.* **B 210** (1982) 181;
 J.M. Evans, D. Kagan and C.A.S. Young, *Non-local charges and quantum integrability of σ-models on the symmetric spaces SO(2n)/SO(n) × SO(n) and Sp(2n)/Sp(n) × Sp(n)*, *Phys. Lett.* **B 597** (2004) 112 [[hep-th/0404003](#)].
- [17] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [[hep-th/0610251](#)].
- [18] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in AdS₅ × S⁵*, *JHEP* **06** (2002) 007 [[hep-th/0204226](#)].
- [19] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in AdS₅ × S⁵*, *Nucl. Phys.* **B 668** (2003) 77 [[hep-th/0304255](#)];
 S.A. Frolov, I.Y. Park and A.A. Tseytlin, *On one-loop correction to energy of spinning strings in S⁵*, *Phys. Rev.* **D 71** (2005) 026006 [[hep-th/0408187](#)].
- [20] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [[hep-th/0204051](#)].
- [21] A.V. Belitsky, A.S. Gorsky and G.P. Korchemsky, *Logarithmic scaling in gauge/string correspondence*, *Nucl. Phys.* **B 748** (2006) 24 [[hep-th/0601112](#)].
- [22] M.K. Benna, S. Benvenuti, I.R. Klebanov and A. Scardicchio, *A test of the AdS/CFT correspondence using high-spin operators*, *Phys. Rev. Lett.* **98** (2007) 131603 [[hep-th/0611135](#)];
 L.F. Alday, G. Arutyunov, M.K. Benna, B. Eden and I.R. Klebanov, *On the strong coupling scaling dimension of high spin operators*, *JHEP* **04** (2007) 082 [[hep-th/0702028](#)];
 I. Kostov, D. Serban and D. Volin, *Strong coupling limit of Bethe ansatz equations*, *Nucl. Phys.* **B 789** (2008) 413 [[hep-th/0703031](#)];
 P.Y. Casteill and C. Kristjansen, *The strong coupling limit of the scaling function from the quantum string Bethe ansatz*, *Nucl. Phys.* **B 785** (2007) 1 [[arXiv:0705.0890](#)];
 B. Basso, G.P. Korchemsky and J. Kotanski, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev. Lett.* **100** (2008) 091601 [[arXiv:0708.3933](#)].
- [23] S. Frolov, A. Tirziu and A.A. Tseytlin, *Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT*, *Nucl. Phys.* **B 766** (2007) 232 [[hep-th/0611269](#)].

- [24] L. Freyhult, A. Rej and M. Staudacher, *A generalized scaling function for AdS/CFT*, *J. Stat. Mech.* (2008) P0707015 [[arXiv:0712.2743](#)].
- [25] D. Fioravanti, P. Grinza and M. Rossi, *Strong coupling for planar $\mathcal{N} = 4$ SYM theory: an all-order result*, [arXiv:0804.2893](#); *The generalised scaling function: a note*, [arXiv:0805.4407](#);
D. Bombardelli, D. Fioravanti and M. Rossi, *Large spin corrections in $\mathcal{N} = 4$ SYM $sl(2)$: still a linear integral equation*, [arXiv:0802.0027](#).
- [26] L.F. Alday and J.M. Maldacena, *Comments on operators with large spin*, *JHEP* **11** (2007) 019 [[arXiv:0708.0672](#)].
- [27] T. McLoughlin and R. Roiban, *Spinning strings at one-loop in $AdS_4 \times P^3$* , [arXiv:0807.3965](#).
- [28] C. Krishnan, *AdS_4/CFT_3 at one loop*, *JHEP* **09** (2008) 092 [[arXiv:0807.4561](#)].
- [29] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018 [[hep-th/0512253](#)].
- [30] L.F. Alday, G. Arutyunov and S. Frolov, *New integrable system of 2Ds fermions from strings on $AdS_5 \times S^5$* , *JHEP* **01** (2006) 078 [[hep-th/0508140](#)];
G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring*, *J. Phys. A* **40** (2007) 3583 [[hep-th/0609157](#)].
- [31] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$* , *JHEP* **02** (2005) 059 [[hep-th/0411089](#)];
S. Frolov, J. Plefka and M. Zamaklar, *The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*, *J. Phys. A* **39** (2006) 13037 [[hep-th/0603008](#)].